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On a free boundary problem for the Reynolds equation derived from the Stokes system with Tresca boundary conditions

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Abstract

The asymptotic behaviour of a Stokes flow with Tresca free boundary friction conditions when one dimension of the fluid domain tends to zero is studied. A specific Reynolds equation associated with variational inequalities is obtained and uniqueness is proved.
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1. Introduction

Solving fluid equations like (Navier) Stokes one requires the knowledge of the velocities on the fluid–solid interface. This subject is often a matter of discussion as a lot of physical parameters are involved like micro-roughness of the surface or the rheological properties of the fluid. No-slip condition in which the fluid is assumed to have the same velocity as the surrounding solid boundary is widely used in mathematical studies [16]. Nevertheless, this boundary condition is sometimes overlooked and it is possible to deal with the "slip condition" which allows the fluid to slip on the surface but not to go through it. The nor-

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mal component of the velocity is equal to zero while the tangential one is proportional to the tangential stresses. Existence and uniqueness theorems for a weak related formulation are easily obtained (see, for example, [1]). The intermediate case in which the slip condition only occurs for sufficiently large ratio between tangential stresses and normal stresses while the no-slip condition is retained for small ratio have also been introduced in [7] and an existence theorem obtained for a non-Newtonian fluid. This last case is nothing else than a transposition of the well-known Coulomb law between two solids [8] to the fluid solid interface and so leads to a free boundary problem model. An accurate choice of these boundary conditions is of particular interest in the lubrication area which is concerned with thin film flow behaviour. In this case, the difference of velocities between the surrounding surfaces is the governing phenomena that allows the pressure in the fluid to build up and prevent the solid surfaces from being in contact which is the main objective of the lubrication. Continuous experimental studies are being conducted [13] but are still difficult due to the thickness of the gap between the solid surfaces which can be as small as 50 nanometers. In such operating conditions, a no-slip condition is induced by chemical bonds between the lubricant and the surrounding surfaces. Conversely, tangential stresses are so high that they tend to destroy the chemical bonds and induce a slip phenomena. Such behaviour is then close to the Tresca free boundary friction model in solid mechanics [9]. This phenomenon has been related in a lot of mechanical papers for both Newtonian and non-Newtonian cases [12,17]. Although being implicitly used in numerical procedures in lubrication problems, a Reynolds thin film equation taking account of such slip phenomena seems to have been posed for the first time in a somewhat mathematical aspect in [15]. This study is restricted to one-dimensional problems and the existence of the discretized problem is proved. The aim of this paper is not only to give existence and uniqueness for this problem but also to obtain rigorously the equation describing such phenomena in a thin film flow by way of an asymptotic analysis in which the small parameter is the width of the gap, following the same ideas as in [2–5]. The departure point is the Stokes equation with the Tresca boundary conditions and so fall into the scope of the work of [7]. Nevertheless, using the Tresca condition instead of the Coulomb ones allows us not only to get an existence theorem but also a uniqueness one.

2. Basic equations and assumptions

Let ω be a fixed bounded domain of $\mathbf{R}^2$ plane ($x = (x_1, x_2)$). We suppose that ω has a Lipschitz continuous boundary and is the bottom of the fluid domain. The upper surface Γ_1^ε is defined by $x_3 = H(x) = H(x_1, x_2)$. Assuming that the fluid film between the surfaces is thin, we introduce a small parameter $\varepsilon \leqslant 1$, that will tend to zero, and a positive smooth and bounded function h such that $H(x) = \varepsilon h(x)$. We denote

$$\Omega^\varepsilon = \{(x, x_3) \in \mathbf{R}^3\colon\ x \in \omega \text{ and } 0 < x_3 < \varepsilon h(x)\}.$$

Let Γ^ε be the boundary of Ω^ε. We have $\Gamma^\varepsilon = \bar{\omega} \cup \bar{\Gamma}_1^\varepsilon \cup \bar{\Gamma}_L^\varepsilon$, where Γ_L^ε is the lateral boundary. For given f^ε in $(L^2(\Omega^\varepsilon))^3$, the motion in the fluid is described by the basic Stokes system of equations

$$-\nu \Delta u^\varepsilon + \nabla p^\varepsilon = f^\varepsilon \quad \text{in } \Omega^\varepsilon, \tag{2.1}$$

where p^ε, u^ε, and ν are the pressure, the velocity field, the viscosity, and f^ε the body forces; the incompressibility equation

$$\operatorname{div}(u^\varepsilon) = 0 \quad \text{in } \Omega^\varepsilon. \tag{2.2}$$

To describe the boundary conditions, let us introduce first a function g in $(H^{1/2}(\Gamma^\varepsilon))^3$ such that

$$\int_{\Gamma^\varepsilon} g \cdot n \, d\sigma = 0, \quad g_3 = 0 \quad \text{on } \Gamma_L^\varepsilon, \qquad g = 0 \quad \text{on } \Gamma_1^\varepsilon, \qquad g \cdot n = 0 \quad \text{on } \omega, \tag{2.3}$$

where $n = (n_1, n_2, n_3)$ is the outward unit normal to Γ^ε.

The actual velocities on the boundary are

- On Γ_1^ε, no-slip condition is given. The upper surface is assumed to be fixed as
$$u^\varepsilon = 0. \tag{2.4}$$
- On Γ_L^ε, the velocity is known and chosen parallel to the ω-plane,
$$u^\varepsilon = g. \tag{2.5}$$
- On ω, there is a no-flux condition across ω so that
$$u^\varepsilon \cdot n = 0. \tag{2.6}$$

The tangential velocity is unknown and satisfies the Tresca friction law with k^ε upper limit for the stress

$$\begin{cases} |\sigma_T^\varepsilon| = k^\varepsilon \quad \Rightarrow \quad \exists \lambda \geqslant 0,\ u_T^\varepsilon = s - \lambda \sigma_T^\varepsilon, \\ |\sigma_T^\varepsilon| < k^\varepsilon \quad \Rightarrow \quad u_T^\varepsilon = s, \end{cases} \tag{2.7}$$

where $|\cdot|$ denotes the $\mathbf{R}^2$ Euclidean norm, s is the velocity of the lower surface ω, σ_n^ε and σ_T^ε are, respectively, the components of the normal and the tangential stress tensor

$$\sigma_{ij}^\varepsilon = -p^\varepsilon \delta_{ij} + \nu \left(\frac{\partial u_i^\varepsilon}{\partial x_j} + \frac{\partial u_j^\varepsilon}{\partial x_i} \right) \quad (1 \leqslant i, j \leqslant 3),$$

where $\sigma_n^\varepsilon = \sigma_{ij}^\varepsilon n_i n_j = (\sigma^\varepsilon \cdot n) \cdot n$, $\sigma_{T_i}^\varepsilon = \sigma_{ij}^\varepsilon n_j - \sigma_n^\varepsilon n_i$, and u_T^ε is the tangential velocity $u_{T_i}^\varepsilon = u_i^\varepsilon - u_j^\varepsilon n_j n_i$.

Due to (2.3) and following [11, Lemma 2.2, p. 24], there exists function G^ε in $(H^1(\Omega^\varepsilon))^3$ such that

$$\operatorname{div}(G^\varepsilon) = 0 \quad \text{in } \Omega^\varepsilon, \qquad G^\varepsilon = g \quad \text{on } \Gamma^\varepsilon. \tag{2.8}$$

To get a weak formulation, we introduce

$$V(\Omega^\varepsilon) = \left\{ v \in \left(H^1(\Omega^\varepsilon)\right)^3:\ v \cdot n = 0 \text{ on } \omega,\ v = 0 \text{ on } \Gamma_L^\varepsilon \cup \Gamma_1^\varepsilon \right\},$$
$$K^\varepsilon = \left\{ \varphi \in \left(H^1(\Omega^\varepsilon)\right)^3:\ \varphi \cdot n = 0 \text{ on } \omega,\ \varphi = G^\varepsilon \text{ on } \Gamma_L^\varepsilon \cup \Gamma_1^\varepsilon \right\},$$
$$L_0^2(\Omega^\varepsilon) = \left\{ q \in L^2(\Omega^\varepsilon):\ \int_{\Omega^\varepsilon} q \, dx \, dx_3 = 0,\ \text{where } dx = dx_1 \, dx_2 \right\}.$$

A formal application of Green's formula, using (2.1)–(2.7) leads to the weak formulation:

For G^ε as in (2.8), find $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon, u_3^\varepsilon)$ in K^ε and p^ε in $L_0^2(\Omega^\varepsilon)$, such that

$$\int_{\Omega^\varepsilon} q \operatorname{div}(u^\varepsilon)\, dx\, dx_3 = 0, \quad \forall q \in L_0^2(\Omega^\varepsilon), \tag{2.9}$$

$$\sum_{i,j=1}^{3} \int_{\Omega^\varepsilon} \left(\nu \left(\frac{\partial u_i^\varepsilon}{\partial x_j} + \frac{\partial u_j^\varepsilon}{\partial x_i} \right) - p^\varepsilon \delta_{i,j} \right) \frac{\partial}{\partial x_j} \left(\varphi_i - u_i^\varepsilon \right) dx\, dx_3 + \int_\omega k^\varepsilon |\varphi - s|\, dx$$
$$- \int_\omega k^\varepsilon |u^\varepsilon - s|\, dx \geqslant \sum_{i=1}^{3} \int_{\Omega^\varepsilon} f_i^\varepsilon \left(\varphi_i - u_i^\varepsilon \right) dx\, dx_3, \quad \forall \varphi \in K^\varepsilon. \tag{2.10}$$

Remark 1. Using an idea in [7], we give in the following theorem a proof of existence and uniqueness to (2.9)–(2.10).

Theorem 2.1. *Assume that f^ε in $(L^2(\Omega^\varepsilon))^3$ and the friction coefficient k^ε is a nonnegative function in $L^\infty(\omega)$; then there exists a unique u^ε and there exists a unique* (*up to an additive constant*) *p^ε such that $(u^\varepsilon, p^\varepsilon)$ in $K^\varepsilon \times L_0^2(\Omega^\varepsilon)$ is a solution to problem* (2.9)–(2.10).

Proof. From (2.10) and taking in mind (2.9) we get that u^ε satisfies the following variational problem:

Find u in K^ε such that $\operatorname{div}(u) = 0$, and

$$a(u, \varphi - u) + j(\varphi) - j(u) \geqslant (f^\varepsilon, \varphi - u), \quad \forall \varphi \in K^\varepsilon,\ \operatorname{div}(\varphi) = 0, \tag{2.11}$$

where

$$a(u, \varphi) = \sum_{i,j=1}^{3} \int_{\Omega^\varepsilon} \frac{\nu}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \left(\frac{\partial \varphi_i}{\partial x_j} + \frac{\partial \varphi_j}{\partial x_i} \right) dx\, dx_3,$$

$$(f^\varepsilon, \varphi) = \sum_{i=1}^{3} \int_{\Omega^\varepsilon} f_i^\varepsilon \varphi_i\, dx\, dx_3, \qquad j(\varphi) = \int_\omega k^\varepsilon |\varphi - s|\, dx.$$

The bilinear form $a(\cdot,\cdot)$ is continuous on $V(\Omega^\varepsilon) \times V(\Omega^\varepsilon)$, and using the Korn's inequality we deduce that $a(\cdot,\cdot)$ is coercive on $V(\Omega^\varepsilon) \times V(\Omega^\varepsilon)$, moreover j is a convex and continuous functional on $V(\Omega^\varepsilon)$. Then the existence and uniqueness of u^ε in K^ε satisfying the variational inequality of the second kind (2.11) is well known and follows, for example, from [6,14].

To get p^ε, we will apply the duality results of convex optimisation [10, Theorem 4.1, p. 58 and Remark 4.2, pp. 59–61]. For this, notice first that we can rewrite (2.11) so that it is defined on the whole of $V(\Omega^\varepsilon)$ by introducing the indicator functions

$$\psi_{K^\varepsilon} : \left(L^2(\Omega^\varepsilon)\right)^3 \to \bar{\mathbf{R}} \quad \text{such that } \varphi \mapsto \psi_{K^\varepsilon}(\varphi) = \begin{cases} 0 & \text{if } \varphi \in K^\varepsilon, \\ +\infty & \text{if } \varphi \notin K^\varepsilon, \end{cases}$$

and

$$\mathcal{H}: L^2(\Omega^\varepsilon) \to \bar{\mathbf{R}} \quad \text{such that } q \mapsto \mathcal{H}(q) = \begin{cases} 0 & \text{if } q = 0, \\ +\infty & \text{if } q \neq 0, \end{cases}$$

so (2.11) is equivalent to

$$\begin{aligned} &a(u, \varphi - u) + j(\varphi) - j(u) + \psi_{K^\varepsilon}(\varphi) - \psi_{K^\varepsilon}(u) \geqslant (f^\varepsilon, \varphi - u), \\ &\quad \forall \varphi \in V(\Omega^\varepsilon),\ \operatorname{div}(\varphi) = 0, \end{aligned} \tag{2.12}$$

and the unique solution of (2.11) minimizes the functional

$$\inf_{\varphi \in V(\Omega^\varepsilon)} \left\{ \frac{1}{2} a(\varphi, \varphi) - (f^\varepsilon, \varphi) + j(\varphi) + \mathcal{H}\big(\operatorname{div}(\varphi)\big) + \psi_{K^\varepsilon}(\varphi) \right\}, \tag{2.13}$$

which can be writen in the following form:

$$\inf_{\varphi \in V(\Omega^\varepsilon)} F(\varphi) + \mathcal{G}\big(\Lambda(\varphi)\big),$$

where

$$\begin{aligned} &F : V(\Omega^\varepsilon) \to \mathbf{R} \quad \text{such that } \varphi \mapsto F(\varphi) = \frac{1}{2} a(\varphi, \varphi) - (f, \varphi), \\ &\Lambda : V(\Omega^\varepsilon) \to Y = L^2(\omega) \times L^2(\Omega^\varepsilon) \times V(\Omega^\varepsilon) \\ &\quad \text{such that } \varphi \mapsto \Lambda(\varphi) = (\Lambda_1 \varphi, \Lambda_2 \varphi, \varphi) = (\varphi|_\omega, \operatorname{div}(\varphi), \varphi), \\ &\mathcal{G} : Y \to \bar{\mathbf{R}} \quad \text{such that } q \mapsto \mathcal{G}(q) = j(q_1) + \mathcal{H}(q_2) + \psi_{K^\varepsilon}(q_3). \end{aligned}$$

Then, the dual problem (to (2.13)) is given by:

$$\begin{aligned} &\text{Find } p^\star \text{ in } Y^\star = L^2(\omega) \times L^2(\Omega^\varepsilon) \times V^\star(\Omega^\varepsilon) \text{ solution of the problem} \\ &\sup_{q^\star \in Y^\star} \big\{ -F^\star(\Lambda^\star q^\star) - \mathcal{G}^\star(-q^\star) \big\}, \end{aligned} \tag{2.14}$$

where

$$F^\star(\Lambda^\star q^\star) = \sup_{\varphi \in V(\Omega^\varepsilon)} \big\{ \langle \Lambda_1^\star q_1^\star, \varphi \rangle + \langle \Lambda_2^\star q_2^\star, \varphi \rangle + \langle \Lambda_3^\star q_3^\star, \varphi \rangle - F(\varphi) \big\},$$

$$\begin{aligned} \mathcal{G}^\star(-q^\star) &:= \sup_{q \in Y} \big\{ \langle -q^\star, q \rangle - \mathcal{G}(q) \big\} \\ &= \sup_{q_1 \in L^2(\omega)} \big\{ \langle -q_1^\star, q_1 \rangle - j(q_1) \big\} \\ &\quad + \sup_{q_2 \in L^2(\Omega^\varepsilon)} \big\{ \langle -q_2^\star, q_2 \rangle - \mathcal{H}(q_2) \big\} + \sup_{q_3 \in V(\Omega^\varepsilon)} \big\{ \langle -q_3^\star, q_3 \rangle - \psi_{K^\varepsilon}(q_3) \big\}, \end{aligned}$$

and from the definition of $\mathcal{H}$, we have for any $q = (q_1, q_2, q_3)$ in $Y = L^2(\omega) \times L^2(\Omega^\varepsilon) \times V(\Omega^\varepsilon)$,

$$\mathcal{G}^\star(-q^\star) \geqslant \big\{ \langle -q_1^\star, q_1 \rangle - j(q_1) \big\} + \big\{ \langle -q_3^\star, q_3 \rangle - \psi_{K^\varepsilon}(q_3) \big\}.$$

As the function $\mathcal{G}^\star$ from $Y^\star \to \mathbf{R}$, is continuous, the hypothesis of [10, Chapter III, Theorem 4.1] are satisfied for the dual problem (2.14), and imply the existence of $p^\star$ in $Y^\star$ satisfying

$$\left\{F(u^\varepsilon) + \mathcal{G}\big(\Lambda(u^\varepsilon)\big)\right\} + \left\{F^\star(\Lambda^\star p^\star) + \mathcal{G}^\star(-p^\star)\right\} = 0,$$

which can be written as

$$\begin{aligned}&\left\{F(u^\varepsilon) + j(\Lambda_1 u^\varepsilon) + \mathcal{H}(\Lambda_2 u^\varepsilon) + \psi_{K^\varepsilon}(\Lambda_3 u^\varepsilon)\right\}\\ &\quad + \left\{F^\star(\Lambda^\star p^\star) + j^\star\big(-p_1^\star\big) + (\psi_{K^\varepsilon})^\star\big(-p_3^\star\big)\right\} = 0.\end{aligned}$$

Let us remark from the definition of $\mathcal{H}$ and by choosing $q = \Lambda\varphi$ for any φ in V^ε that

$$\begin{aligned}&F(u^\varepsilon) - F(\varphi) + j(\Lambda_1 u^\varepsilon) - j(\Lambda_1\varphi) + \psi_{K^\varepsilon}(\Lambda_3 u^\varepsilon)\\ &\qquad - \psi_{K^\varepsilon}(\Lambda_3\varphi) + \left\langle p_2^\star, \Lambda_2\varphi\right\rangle - \left\langle p_2^\star, \Lambda_2 u^\varepsilon\right\rangle\\ &\quad \leqslant \left\{-\mathcal{H}(\Lambda_2 u^\varepsilon) - \left\langle p_2^\star \operatorname{div}(u^\varepsilon)\right\rangle\right\} \leqslant 0,\end{aligned}$$

which is exactly

$$\begin{aligned}&a(u^\varepsilon, \varphi - u^\varepsilon) + j(\varphi) - j(u^\varepsilon) + \psi_{K^\varepsilon}(\Lambda_3\varphi) - \psi_{K^\varepsilon}(\Lambda_3 u^\varepsilon) - \left\langle p_2^\star, \operatorname{div}(\varphi - u^\varepsilon)\right\rangle\\ &\quad \geqslant (f^\varepsilon, \varphi - u^\varepsilon), \quad \forall\varphi \in V^\varepsilon.\end{aligned} \tag{2.15}$$

So taking in (2.15) φ and u^ε in K^ε, we get exactly (2.10).

Using Green's formula with $\varphi = u^\varepsilon \pm \phi$ for any ϕ in $(H_0^1(\Omega^\varepsilon))^3$, (2.15) induces

$$\nabla p_2^\star = \nu\Delta u^\varepsilon + f^\varepsilon \quad \text{a.e. in } \Omega^\varepsilon,$$

then as u^ε is unique in K^ε, we deduce the uniqueness (up to an additive constant) of $p_2^\star$ in $L^2(\Omega^\varepsilon)$. □

3. Study of convergence of $(U^\varepsilon, P^\varepsilon)$

According to the change of variables $y = x_3/\varepsilon$, we define the fixed domain

$$\Omega = \left\{(x, y) \text{ such that } x \in \omega, \text{ and } 0 < y < h(x)\right\},$$

and we denote by $\Gamma = \bar{\omega} \cup \bar{\Gamma}_L \cup \bar{\Gamma}_1$ its boundary. We define the following functions in Ω:

$$\hat{u}_i^\varepsilon(x, y) = u_i^\varepsilon(x, x_3) \quad (1 \leqslant i \leqslant 2), \qquad \hat{u}_3^\varepsilon(x, y) = \frac{1}{\varepsilon}u_3^\varepsilon(x, x_3),$$

$$\hat{p}^\varepsilon(x, y) = \varepsilon^2 p^\varepsilon(x, x_3).$$

Let us define first the ε-independent vector

$$\hat{f}(x, y) = \big(\hat{f}_1(x, y), \hat{f}_2(x, y), \hat{f}_3(x, y)\big),$$

then assume the following dependence (with respect to ε) of the data:

$$\hat{f}(x, y) = \varepsilon^2 f^\varepsilon(x, x_3), \qquad \hat{g}(x, y) = g(x, x_3), \quad \text{and} \quad \hat{k} = \varepsilon k^\varepsilon. \tag{3.1}$$

The first assumption means that the body forces cannot be too big while the third one means that k^ε, the upper limit for the tangential stress has the same order of magnitude as the actual stress inside the fluid, which is the ratio of the tangential velocity and of the gap $s/\varepsilon h$.

Let us define, the ε-independent vector $\hat{G}(x,y) = (\hat{G}_1(x,y), \hat{G}_2(x,y), \hat{G}_3(x,y))$ such that

$$\frac{\partial \hat{G}_1}{\partial x_1} + \frac{\partial \hat{G}_2}{\partial x_2} + \frac{\partial \hat{G}_3}{\partial y} = 0 \quad \text{in } \Omega, \qquad \hat{G} = \hat{g} \quad \text{on } \Gamma,$$

and recalling that $g_3 = 0$ on Γ_L, we can choose as G^ε the lift defined by $G_i^\varepsilon(x, x_3) = \hat{G}_i(x,y)$ for $i = 1, 2$ and $G_3^\varepsilon(x, x_3) = \varepsilon \hat{G}_3(x,y)$. We will denote

$$V(\Omega) = \left\{ v \in \left(H^1(\Omega)\right)^3\colon\ v \cdot n = 0 \text{ on } \omega,\ v = 0 \text{ on } \Gamma_L \cup \Gamma_1 \right\},$$

$$K = \left\{ \varphi \in \left(H^1(\Omega)\right)^3\colon\ \varphi \cdot n = 0 \text{ on } \omega,\ v = \hat{G} \text{ on } \Gamma_L \cup \Gamma_1 \right\},$$

$$L_0^2(\Omega) = \left\{ q \in L^2(\Omega)\colon \int_\Omega q\,dx\,dx_3 = 0,\ \text{where } dx = dx_1\,dx_2 \right\}.$$

Then problem (2.9)–(2.10) leads to the following form.

Assuming (3.1), there exists a unique $\hat{u}^\varepsilon$ in K and $\hat{p}^\varepsilon$ in $L_0^2(\Omega)$, such that

$$\int_\Omega q\,\operatorname{div}(\hat{u}^\varepsilon)\,dx\,dy = 0, \quad \forall q \in L_0^2(\Omega), \tag{3.2}$$

$$\begin{aligned}
&\sum_{i,j=1}^{2} \int_\Omega \left(\varepsilon^2 \nu \left(\frac{\partial \hat{u}_i^\varepsilon}{\partial x_j} + \frac{\partial \hat{u}_j^\varepsilon}{\partial x_i} \right) - \hat{p}^\varepsilon \delta_{i,j} \right) \frac{\partial}{\partial x_j} \left(\varphi_i - \hat{u}_i^\varepsilon \right) dx\,dy \\
&\quad + \sum_{i=1}^{2} \int_\Omega \nu \left(\frac{\partial \hat{u}_i^\varepsilon}{\partial y} + \varepsilon^2 \frac{\partial \hat{u}_3^\varepsilon}{\partial x_i} \right) \frac{\partial}{\partial y} \left(\varphi_i - \hat{u}_i^\varepsilon \right) dx\,dy \\
&\quad + \int_\Omega \left(2\nu\varepsilon^2 \frac{\partial \hat{u}_3^\varepsilon}{\partial y} - \hat{p}^\varepsilon \right) \frac{\partial}{\partial y} \left(\varepsilon^{-1}\varphi_3 - \hat{u}_3^\varepsilon \right) dx\,dy \\
&\quad + \sum_{j=1}^{2} \int_\Omega \varepsilon^2 \nu \left(\varepsilon^2 \frac{\partial \hat{u}_3^\varepsilon}{\partial x_j} + \frac{\partial \hat{u}_j^\varepsilon}{\partial y} \right) \frac{\partial}{\partial x_j} \left(\varepsilon^{-1}\varphi_3 - \hat{u}_3^\varepsilon \right) dx\,dy \\
&\geqslant \sum_{i=1}^{2} \int_\Omega \hat{f}_i \left(\varphi_i - \hat{u}_i^\varepsilon \right) dx\,dy + \int_\Omega \varepsilon \hat{f}_3 \left(\varepsilon^{-1}\varphi_3 - \hat{u}_3^\varepsilon \right) dx\,dy \\
&\quad + \int_\omega \hat{k} \left(|\varphi - s| - |\hat{u}^\varepsilon - s| \right) dx, \quad \forall \varphi \in K.
\end{aligned} \tag{3.3}$$

Theorem 3.1. *Assuming* (3.1) *the following estimate on* $\hat{u}^\varepsilon$ *is satisfied*:

$$\frac{\nu\varepsilon^2}{2}\left\|\frac{\partial\hat{u}_1^\varepsilon}{\partial x_1}\right\|^2+\frac{\nu\varepsilon^2}{2}\left\|\frac{\partial\hat{u}_1^\varepsilon}{\partial x_2}\right\|^2+\frac{\nu\varepsilon^2}{2}\left\|\frac{\partial\hat{u}_2^\varepsilon}{\partial x_1}\right\|^2+\left(\frac{\nu}{2}-\frac{\delta^2}{4}\right)\left\|\frac{\partial\hat{u}_1^\varepsilon}{\partial y}\right\|^2$$
$$+\left(\frac{\nu}{2}-\frac{\delta^2}{4}\right)\left\|\frac{\partial\hat{u}_2^\varepsilon}{\partial y}\right\|^2+\frac{\nu\varepsilon^2}{2}\left\|\frac{\partial\hat{u}_2^\varepsilon}{\partial x_2}\right\|^2+\varepsilon^2\left(\frac{\nu}{2}-\frac{\delta^2}{4}\right)\left\|\frac{\partial\hat{u}_3^\varepsilon}{\partial y}\right\|^2$$
$$+\frac{\nu\varepsilon^4}{2}\left\|\frac{\partial\hat{u}_3^\varepsilon}{\partial x_1}\right\|^2+\frac{\nu\varepsilon^4}{2}\left\|\frac{\partial\hat{u}_3^\varepsilon}{\partial x_2}\right\|^2\leqslant C_0, \tag{3.4}$$

where $\|\cdot\|$ *denotes the* L^2*-norm in* Ω, δ *is the diameter of* Ω, *and* C_0 *is an independent constant of* ε.

Proof. Putting $\varphi_i=\hat{G}_i$ for $i=1,2$ and $\varphi_3=\varepsilon\hat{G}_3$, in (3.3), leads to

$$\sum_{i,j=1}^{2}\int_\Omega\left(\varepsilon^2\nu\left(\frac{\partial\hat{u}_i^\varepsilon}{\partial x_j}+\frac{\partial\hat{u}_j^\varepsilon}{\partial x_i}\right)-\hat{p}^\varepsilon\delta_{i,j}\right)\frac{\partial\hat{u}_i^\varepsilon}{\partial x_j}\,dx\,dy+\int_\Omega\left(2\nu\varepsilon^2\frac{\partial\hat{u}_3^\varepsilon}{\partial y}-\hat{p}^\varepsilon\right)\frac{\partial\hat{u}_3^\varepsilon}{\partial y}\,dx\,dy$$
$$+\sum_{i=1}^{2}\int_\Omega\nu\left(\frac{\partial\hat{u}_i^\varepsilon}{\partial y}+\varepsilon^2\frac{\partial\hat{u}_3^\varepsilon}{\partial x_i}\right)\frac{\partial\hat{u}_i^\varepsilon}{\partial y}\,dx\,dy+\sum_{j=1}^{2}\int_\Omega\varepsilon^2\nu\left(\varepsilon^2\frac{\partial\hat{u}_3^\varepsilon}{\partial x_j}+\frac{\partial\hat{u}_j^\varepsilon}{\partial y}\right)\frac{\partial\hat{u}_3^\varepsilon}{\partial x_j}\,dx\,dy$$
$$\leqslant\sum_{i,j=1}^{2}\int_\Omega\left(\varepsilon^2\nu\left(\frac{\partial\hat{u}_i^\varepsilon}{\partial x_j}+\frac{\partial\hat{u}_j^\varepsilon}{\partial x_i}\right)-\hat{p}^\varepsilon\delta_{i,j}\right)\frac{\partial\hat{G}_i}{\partial x_j}\,dx\,dy$$
$$+\int_\Omega\left(2\nu\varepsilon^2\frac{\partial\hat{u}_3^\varepsilon}{\partial y}-\hat{p}^\varepsilon\right)\frac{\partial\hat{G}_3}{\partial y}\,dx\,dy$$
$$+\sum_{i=1}^{2}\int_\Omega\nu\left(\frac{\partial\hat{u}_i^\varepsilon}{\partial y}+\varepsilon^2\frac{\partial\hat{u}_3^\varepsilon}{\partial x_i}\right)\frac{\partial\hat{G}_i}{\partial y}\,dx\,dy+\sum_{j=1}^{2}\int_\Omega\varepsilon^2\nu\left(\varepsilon^2\frac{\partial\hat{u}_3^\varepsilon}{\partial x_j}+\frac{\partial\hat{u}_j^\varepsilon}{\partial y}\right)\frac{\partial\hat{G}_3}{\partial x_j}\,dx\,dy$$
$$+\int_\omega\hat{k}|\hat{G}-s|\,dx-\sum_{i=1}^{2}\int_\Omega\hat{f}_i\big(\hat{G}_i-\hat{u}_i^\varepsilon\big)\,dx\,dy-\int_\Omega\varepsilon\hat{f}_3\big(\hat{G}_3-\hat{u}_3^\varepsilon\big)\,dx\,dy, \tag{3.5}$$

as $\hat{k}$ is positive.

Using (2.9), the Poincaré inequality, $\varepsilon\leqslant 1$, and $2ab\leqslant a^2+b^2$, we deduce

$$\frac{\nu\varepsilon^2}{2}\left\|\frac{\partial\hat{u}_1^\varepsilon}{\partial x_1}\right\|^2+\frac{\nu\varepsilon^2}{2}\left\|\frac{\partial\hat{u}_1^\varepsilon}{\partial x_2}\right\|^2+\frac{\nu\varepsilon^2}{2}\left\|\frac{\partial\hat{u}_2^\varepsilon}{\partial x_1}\right\|^2+\left(\frac{\nu}{2}-\frac{\delta^2}{4}\right)\left\|\frac{\partial\hat{u}_1^\varepsilon}{\partial y}\right\|^2+\left(\frac{\nu}{2}-\frac{\delta^2}{4}\right)\left\|\frac{\partial\hat{u}_2^\varepsilon}{\partial y}\right\|^2$$
$$+\frac{\nu\varepsilon^2}{2}\left\|\frac{\partial\hat{u}_2^\varepsilon}{\partial x_2}\right\|^2+\varepsilon^2\left(\frac{\nu}{2}-\frac{\delta^2}{4}\right)\left\|\frac{\partial\hat{u}_3^\varepsilon}{\partial y}\right\|^2+\frac{\nu\varepsilon^4}{2}\left\|\frac{\partial\hat{u}_3^\varepsilon}{\partial x_1}\right\|^2+\frac{\nu\varepsilon^4}{2}\left\|\frac{\partial\hat{u}_3^\varepsilon}{\partial x_2}\right\|^2$$
$$\leqslant\nu\left\|\frac{\partial\hat{G}_1}{\partial x_1}\right\|^2+\nu\left\|\frac{\partial\hat{G}_1}{\partial x_2}\right\|^2+\nu\left\|\frac{\partial\hat{G}_2}{\partial x_1}\right\|^2+\nu\left\|\frac{\partial\hat{G}_1}{\partial y}\right\|^2+\nu\left\|\frac{\partial\hat{G}_2}{\partial y}\right\|^2$$

$$+\nu\left\|\frac{\partial\hat{G}_2}{\partial x_2}\right\|^2+\nu\left\|\frac{\partial\hat{G}_3}{\partial y}\right\|^2+\nu\left\|\frac{\partial\hat{G}_3}{\partial x_1}\right\|^2+\nu\left\|\frac{\partial\hat{G}_3}{\partial x_2}\right\|^2$$
$$+\|\hat{f}_1\|\|\hat{G}_1\|+\|\hat{f}_2\|\|\hat{G}_2\|+\|\hat{f}_3\|\|\hat{G}_3\|+\left(\|\hat{f}_1\|^2+\|\hat{f}_2\|^2+\|\hat{f}_3\|^2\right)$$
$$+\,\mathrm{const}\|\hat{k}\|_{L^\infty(\omega)}=C_0,$$

thus (3.4) follows. □

Remark 2. If we assume that the body forces $\hat{f}=0$, estimate (3.4) becomes

$$\varepsilon^2\sum_{i,j=1}^{2}\left\|\frac{\partial\hat{u}_i^\varepsilon}{\partial x_j}\right\|^2+\left\|\frac{\partial\hat{u}_1^\varepsilon}{\partial y}\right\|^2+\left\|\frac{\partial\hat{u}_2^\varepsilon}{\partial y}\right\|^2+\varepsilon^2\left\|\frac{\partial\hat{u}_3^\varepsilon}{\partial y}\right\|^2+\varepsilon^4\left\|\frac{\partial\hat{u}_3^\varepsilon}{\partial x_1}\right\|^2+\varepsilon^4\left\|\frac{\partial\hat{u}_3^\varepsilon}{\partial x_2}\right\|^2\leqslant C.$$

Theorem 3.2. *Assuming* (3.1) *and* $\nu>\delta^2/2$ *or* $f=0$, *the following estimates on* p^ε *are satisfied*:

$$\left\|\frac{\partial\hat{p}^\varepsilon}{\partial x_i}\right\|_{H^{-1}(\Omega)}\leqslant C_1\quad(i=1,2),\tag{3.6}$$
$$\left\|\frac{\partial\hat{p}^\varepsilon}{\partial y}\right\|_{H^{-1}(\Omega)}\leqslant\varepsilon\cdot C_2,\tag{3.7}$$

where C_1 *and* C_2 *denote independent constants of* ε.

Proof. Let ψ in $H_0^1(\Omega)$, putting in (3.3) $\varphi_i=\hat{u}_i^\varepsilon$ (for $i=1,2$) and $\varphi_3=\varepsilon\hat{u}_3^\varepsilon\pm\psi$, we deduce

$$-\int_\Omega\hat{p}^\varepsilon\frac{\partial\psi}{\partial y}\,dx\,dy=-\int_\Omega 2\nu\varepsilon^2\frac{\partial\hat{u}_3^\varepsilon}{\partial y}\frac{\partial\psi}{\partial y}\,dx\,dy$$
$$-\sum_{j=1}^{2}\int_{\Omega^\varepsilon}\varepsilon^2\nu\left(\varepsilon^2\frac{\partial\hat{u}_3^\varepsilon}{\partial x_j}+\frac{\partial\hat{u}_j^\varepsilon}{\partial y}\right)\frac{\partial\psi}{\partial x_j}\,dx\,dy+\int_\Omega\varepsilon\hat{f}_3\psi\,dx\,dy.\tag{3.8}$$

Taking in (3.3) $\varphi_1=\hat{u}_1^\varepsilon\pm\psi$, ψ in $H_0^1(\Omega)$, $\varphi_2=\hat{u}_2^\varepsilon$, $\varphi_3=\hat{u}_3^\varepsilon$, we get

$$-\int_\Omega\hat{p}^\varepsilon\frac{\partial\psi}{\partial x_1}\,dx\,dy=-\int_\Omega 2\varepsilon^2\nu\frac{\partial\hat{u}_1^\varepsilon}{\partial x_1}\frac{\partial\psi}{\partial x_1}\,dx\,dy-\int_\Omega\varepsilon^2\nu\left(\frac{\partial\hat{u}_1^\varepsilon}{\partial x_2}+\frac{\partial\hat{u}_2^\varepsilon}{\partial x_1}\right)\frac{\partial\psi}{\partial x_1}\,dx\,dy$$
$$-\int_\Omega\nu\left(\frac{\partial\hat{u}_1^\varepsilon}{\partial y}+\varepsilon^2\frac{\partial\hat{u}_3^\varepsilon}{\partial x_1}\right)\frac{\partial\psi}{\partial y}\,dx\,dy+\int_\Omega\hat{f}_1\psi\,dx\,dy,$$
$$\forall\psi\in H_0^1(\Omega).\tag{3.9}$$

In the same way, the choice $\varphi_1=\hat{u}_1^\varepsilon$, $\varphi_2=\hat{u}_2^\varepsilon\pm\psi$, ψ in $H_0^1(\Omega)$, $\varphi_3=\varepsilon\hat{u}_3^\varepsilon$, leads to

$$-\int_\Omega \hat{p}^\varepsilon \frac{\partial\psi}{\partial x_2}\,dx\,dy = -\int_\Omega 2\varepsilon^2\nu \frac{\partial \hat{u}_2^\varepsilon}{\partial x_2}\frac{\partial\psi}{\partial x_2}\,dx\,dy - \int_\Omega \varepsilon^2\nu\left(\frac{\partial \hat{u}_1^\varepsilon}{\partial x_2}+\frac{\partial \hat{u}_2^\varepsilon}{\partial x_1}\right)\frac{\partial\psi}{\partial x_1}\,dx\,dy$$
$$-\int_\Omega \nu\left(\frac{\partial \hat{u}_2^\varepsilon}{\partial y}+\varepsilon^2\frac{\partial \hat{u}_3^\varepsilon}{\partial x_2}\right)\frac{\partial\psi}{\partial y}\,dx\,dy + \int_\Omega \hat{f}_2\psi\,dx\,dy,$$
$$\forall\psi \in H_0^1(\Omega), \tag{3.10}$$

then from (3.8) using (3.4) we get (3.7), and from (3.9)–(3.10) using (3.4) we get (3.6). □

Corollary 3.3. *Let the assumptions of Theorems* 2.1 *and* 3.2 *hold*; *then there exists* $u_i^\star$ *in* V_y $(i = 1, 2)$, *and* $p^\star$ *in* $L_0^2(\Omega)$ *such that*

$$\hat{u}_i^\varepsilon \rightharpoonup u_i^\star \quad (1 \leqslant i \leqslant 2) \text{ weakly in } V_y, \tag{3.11}$$

where $V_y = \{\psi \in L^2(\Omega)$ *such that* $\partial\psi/\partial y \in L^2(\Omega)\}$,

$$\varepsilon\frac{\partial \hat{u}_i^\varepsilon}{\partial x_j} \rightharpoonup 0 \quad (1 \leqslant i, j \leqslant 2) \text{ weakly in } L^2(\Omega), \tag{3.12}$$

$$\varepsilon\frac{\partial \hat{u}_3^\varepsilon}{\partial y} \rightharpoonup 0 \quad \text{weakly in } L^2(\Omega), \tag{3.13}$$

$$\varepsilon^2\frac{\partial \hat{u}_3^\varepsilon}{\partial x_i} \rightharpoonup 0 \quad (1 \leqslant i \leqslant 2) \text{ weakly in } L^2(\Omega), \tag{3.14}$$

$$\varepsilon\hat{u}_3^\varepsilon \rightharpoonup 0 \quad \text{weakly in } L^2(\Omega), \tag{3.15}$$

$$\hat{p}^\varepsilon \rightharpoonup p^\star \quad \text{weakly in } L_0^2(\Omega). \tag{3.16}$$

Proof. From (3.4) there exists a fixed constant C which does not depend on ε such that

$$\left\|\frac{\partial \hat{u}_i^\varepsilon}{\partial y}\right\| \leqslant C \quad (1 \leqslant i \leqslant 2).$$

Using the above estimate and the Poincare inequality in the domain Ω we deduce (3.11). Also (3.12)–(3.14) follows from (3.4), and (3.16) follows from (3.6), (3.7), and [16]. To prove (3.15), as in [3] we choose q such that $q(x, y) = y\theta(x) - \gamma$, where θ in $\mathcal{C}_0^\infty(\omega)$ and

$$\gamma = \frac{\int_\Omega y\theta\,dx\,dy}{\int_\Omega dx\,dy}.$$

Using (3.2) and the Green formula, the boundary conditions on Γ imply

$$-\sum_{i=1}^{2}\int_\Omega y\hat{u}_i^\varepsilon \frac{\partial\theta}{\partial x_i}\,dx\,dy - \int_\Omega \theta\hat{u}_3^\varepsilon\,dx\,dy = 0.$$

As $\hat{u}_i^\varepsilon \rightharpoonup u^\star$ in V_y $(i = 1, 2)$, (3.15) holds. □

4. Study of the limit problem

In this section, we give both the equations satisfied by $p^\star$ and $u^\star$ in Ω and the inequalities for the trace of the velocity $u^\star(x,0)$ and the stress $(\partial u^\star/\partial y)(x,0)$ on $\partial\omega$.

Theorem 4.1. *With the same assumptions as Theorem* 3.2, $(u^\star, p^\star)$ *satisfy*

$$p^\star \in H^1(\omega), \tag{4.1}$$

$$-\nu \frac{\partial^2 u_i^\star}{\partial y^2} + \frac{\partial p^\star}{\partial x_i} = \hat{f}_i \quad (i=1,2) \text{ in } L^2(\Omega). \tag{4.2}$$

Proof. We choose in (3.3) $\varphi_3 = \hat{u}_3^\varepsilon \pm \psi$ with ψ in $H_0^1(\Omega)$, we deduce

$$\sum_{j=1}^{2} \int_{\Omega^\varepsilon} \varepsilon^2 \nu \left(\varepsilon^2 \frac{\partial \hat{u}_3^\varepsilon}{\partial x_j} + \frac{\partial \hat{u}_j^\varepsilon}{\partial y} \right) \frac{\partial \psi}{\partial x_j}\, dx\, dy + \int_{\Omega} \left(2\nu\varepsilon^2 \frac{\partial \hat{u}_3^\varepsilon}{\partial y} - \hat{p}^\varepsilon \right) \frac{\partial \psi}{\partial y}\, dx\, dy$$
$$= \int_{\Omega} \varepsilon f_3 \psi\, dx\, dy.$$

Using (3.14), (3.11), (3.13), and the hypothesis of this theorem, we obtain

$$\int_{\Omega} p^\star \frac{\partial \psi_3}{\partial y}\, dx\, dy = 0, \quad \forall \psi \in H_0^1(\Omega),$$

then

$$\frac{\partial p^\star}{\partial y} = 0 \quad \text{in } H^{-1}(\Omega). \tag{4.3}$$

Choosing now $\varphi_i = \hat{u}_i^\varepsilon \pm \psi_i$ (for $i=1,2$) with ψ_i in $H_0^1(\Omega)$ and $\varphi_3 = \varepsilon \hat{u}_3^\varepsilon$, in (3.3), leads to

$$\sum_{i,j=1}^{2} \int_{\Omega} \left(\varepsilon^2 \nu \left(\frac{\partial \hat{u}_i^\varepsilon}{\partial x_j} + \frac{\partial \hat{u}_j^\varepsilon}{\partial x_i} \right) - \hat{p}^\varepsilon \delta_{i,j} \right) \frac{\partial \psi_i}{\partial x_j}\, dx\, dy$$
$$+ \sum_{i=1}^{2} \int_{\Omega} \nu \left(\frac{\partial \hat{u}_i^\varepsilon}{\partial y} + \varepsilon^2 \frac{\partial \hat{u}_3^\varepsilon}{\partial x_i} \right) \frac{\partial \psi_i}{\partial y}\, dx\, dy + \sum_{i=1}^{2} \int_{\Omega} \hat{f}_i \psi_i\, dx\, dy. \tag{4.4}$$

Using (3.12), (3.16), (3.11), (3.14), and the hypothesis of this theorem, we deduce first with $\psi_1 = 0$ and ψ_2 in $H_0^1(\Omega)$, then with $\psi_2 = 0$ and ψ_1 in $H_0^1(\Omega)$, the following equality:

$$-\sum_{i=1}^{2} \int_{\Omega} p^\star \frac{\partial \psi_i}{\partial x_i}\, dx\, dy + \sum_{i=1}^{2} \int_{\Omega} \nu \frac{\partial \hat{u}_i^*}{\partial y} \frac{\partial \psi_i}{\partial y}\, dx\, dy = \sum_{i=1}^{2} \int_{\Omega} \hat{f}_i \psi_i\, dx\, dy, \tag{4.5}$$

then using the Green formula, we obtain

$$-\nu \frac{\partial^2 u_i^\star}{\partial y^2} + \frac{\partial p^\star}{\partial x_i} = \hat{f}_i \quad (i=1,2) \text{ in } H^{-1}(\Omega). \tag{4.6}$$

To prove that $p^\star$ is in $H^1(\omega)$, let us recall first that $p^\star$ is a function of (x_1, x_2) only from (4.3), then following [3] we choose ψ_i in (4.5) such that $\psi_i(x, y) = y(y - h(x))\theta(x)$ with θ in $H_0^1(\omega)$, and using the Green formula we deduce

$$\frac{1}{6}\int_\omega p^\star \frac{\partial(h^3\theta)}{\partial x_i}\,dx - 2\nu\int_\omega h\tilde{u}_i^\star\theta\,dx = \int_\omega \tilde{f}_i\theta\,dx,$$

where

$$\tilde{u}_i^\star(x) = \frac{1}{h(x)}\int_0^{h(x)} u_i^\star(x, y)\,dy \quad \text{and} \quad \tilde{f}_i(x) = \int_0^{h(x)} y\big(y - h(x)\big)\hat{f}_i(x, y)\,dy.$$

Whence

$$2\nu h\tilde{u}_i^\star - \frac{1}{6}h^3\frac{\partial p^\star}{\partial x_i} = \tilde{f}_i \quad (i = 1, 2) \text{ in } H^{-1}(\omega). \tag{4.7}$$

As f_i is in $L^2(\Omega)$, $u_i^\star$ in V_y then in $L^2(\Omega)$, therefore $\tilde{f}_i$ and $\tilde{u}_i^\star$ are in $L^2(\omega)$. Then from (4.7) we get $p^\star$ in $H^1(\omega)$, and (4.1) follows. So as f_i belongs to $L^2(\Omega)$, from (4.6) we have $\partial^2 u_i^\star/\partial y^2$ in $L^2(\Omega)$. Whence (4.2) holds, and we also have $\partial u_i^\star/\partial y$ in V_y. $\square$

For convenience, we will denote by $s^\star(x) = u^\star(x, 0)$ and $\tau^\star(x) = (\partial u^\star/\partial y)(x, 0)$; as $\partial u^\star/\partial y$ in V_y, $\tau^\star$ belongs to $L^2(\omega)$, and we have

Theorem 4.2. *Under the same hypothesis of Theorem* 3.2, $(s^\star, \tau^\star)$ *satisfy the following inequalities*:

$$\int_\omega \hat{k}\big(|\psi + s^\star - s| - |s^\star - s|\big)\,dx - \int_\omega \nu\tau^\star\psi\,dx \geqslant 0, \quad \forall\psi \in \big(L^2(\omega)\big)^2, \tag{4.8}$$

$$\begin{cases} \nu|\tau^\star| = \hat{k} & \Rightarrow \quad \exists\lambda \geqslant 0,\ s^\star = s + \lambda\tau^\star, \\ \nu|\tau^\star| < \hat{k} & \Rightarrow \quad s^\star = s, \quad \textit{a.e. in } \omega, \end{cases} \tag{4.9}$$

where $|\cdot|$ *denotes the* $\mathbf{R}^2$ *Euclidean norm.*

Proof. Choosing $\varphi = (\varphi_1, \varphi_2, \varepsilon\hat{u}_3^\varepsilon)$ with $\varphi_i = \hat{u}_i^\varepsilon + \psi_i$ (for $i = 1, 2$) and ψ_i in $H^1_{\Gamma_1\cap\Gamma_L}(\omega)$, where $H^1_{\Gamma_1\cap\Gamma_L}(\omega) = \{v \in H^1(\Omega)\colon\ v = 0 \text{ on } \Gamma_1 \cap \Gamma_L\}$, in (3.3), leads to

$$\begin{aligned}
&\sum_{i,j=1}^{2}\int_\Omega \left(\varepsilon^2\nu\left(\frac{\partial\hat{u}_i^\varepsilon}{\partial x_j} + \frac{\partial\hat{u}_j^\varepsilon}{\partial x_i}\right) - \hat{p}^\varepsilon\delta_{i,j}\right)\frac{\partial\psi_i}{\partial x_j}\,dx\,dy \\
&\quad + \sum_{i=1}^{2}\int_\Omega \nu\left(\frac{\partial\hat{u}_i^\varepsilon}{\partial y} + \varepsilon^2\frac{\partial\hat{u}_3^\varepsilon}{\partial x_i}\right)\frac{\partial\psi_i}{\partial y}\,dx\,dy \\
&\quad + \int_\omega \hat{k}\big(|\psi + \hat{u}^\varepsilon - s| - |\hat{u}^\varepsilon - s|\big)\,dx \geqslant \sum_{i=1}^{2}\int_\Omega \hat{f}_i\psi_i\,dx\,dy.
\end{aligned} \tag{4.10}$$

Using Corollary 3.3, we can pass to the limit in (4.10), to obtain

$$\sum_{i=1}^{2}\int_{\Omega} -p^\star \frac{\partial \psi_i}{\partial x_i}\,dx\,dy + \sum_{i=1}^{2}\int_{\Omega} \nu \frac{\partial u_i^\star}{\partial y}\frac{\partial \psi_i}{\partial y}\,dx\,dy + \int_{\omega} \hat{k}\big(|\psi + s^\star - s| - |s^\star - s|\big)\,dx$$
$$\geqslant \sum_{i=1}^{2}\int_{\Omega} \hat{f}_i \psi_i\,dx\,dy.$$

Using now the Green formula, equality (4.2), and the fact that $\psi_i = 0$ on $\Gamma_1 \cap \Gamma_L$ and $\cos(n, x_i) = 0$ on ω, we deduce

$$\int_{\omega} \hat{k}\big(|\psi + s^\star - s| - |s^\star - s|\big)\,dx - \int_{\omega} \nu\tau^\star\psi\,dx \geqslant 0, \quad \forall \psi \in \big(H^1_{\Gamma_1\cup\Gamma_L}(\Omega)\big)^2. \tag{4.11}$$

This inequality remains valid for any ψ in $(\mathcal{D}(\omega))^2$ (using the same notations for the trace) and by density of $\mathcal{D}(\omega)$ in $L^2(\omega)$ for any ψ in $(L^2(\omega))^2$. Then (4.8) follows.

To prove (4.9), we take $\psi_i = \pm(s_i^\star - s_i)$, in (4.8), and we obtain

$$\int_{\omega} \big(\hat{k}|s^\star - s| - \nu\tau^\star(s^\star - s)\big)\,dx = 0, \tag{4.12}$$

taking $\psi = \phi - (s^\star - s)$ with ϕ in $(L^2(\omega))^2$, in (4.8), we obtain

$$\int_{\omega} \big(\hat{k}|\phi| - \nu\tau^\star\phi\big)\,dx \geqslant \int_{\omega} \big(\hat{k}|s^\star - s| - \nu\tau^\star(s^\star - s)\big)\,dx.$$

From (4.12) we deduce

$$\int_{\omega} \big(\hat{k}|\phi| - \nu\tau^\star\phi\big)\,dx \geqslant 0, \quad \forall \phi \in \big(L^2(\omega)\big)^2, \tag{4.13}$$

taking first $\phi = (\varphi_1, \varphi_2)$ such that $\varphi_i \geqslant 0$, $i = 1, 2$, in (4.13), we obtain

$$\int_{\omega} \big(\hat{k}|\phi| - \nu|\tau^\star|\cdot|\phi|\cos(\tau^\star, \phi)\big)\,dx = \int_{\omega} \big(\hat{k} - \nu|\tau^\star|\cos(\tau^\star, \phi)\big)|\phi|\,dx \geqslant 0,$$

then

$$\nu|\tau^\star|\cos(\tau^\star, \phi) \leqslant \hat{k} \quad \text{a.e. on } \omega. \tag{4.14}$$

Taking now $-\phi$, with $\phi = (\varphi_1, \varphi_2)$ such that $\varphi_i \geqslant 0$, $i = 1, 2$, in (4.13), we obtain

$$\int_{\omega} \big(\hat{k}|\phi| + \nu|\tau^\star|\cdot|\phi|\cos(\tau^\star, \phi)\big)\,dx = \int_{\omega} \big(\hat{k} + \nu|\tau^\star|\cos(\tau^\star, \phi)\big)|\phi|\,dx \geqslant 0,$$

whence

$$\nu|\tau^\star|\cos(\tau^\star, \phi) \leqslant -\hat{k} \quad \text{a.e. on } \omega. \tag{4.15}$$

From (4.14) and (4.15) we get

$$\nu|\tau^\star| \leqslant \hat{k} \quad \text{a.e. on } \omega, \tag{4.16}$$

then

$$\hat{k}|s^\star - s| \geqslant \nu|\tau^\star| \cdot |s^\star - s| \geqslant \nu\tau^\star \cdot (s^\star - s) \quad \text{a.e. on } \omega,$$
$$\hat{k}|s^\star - s| - \nu\tau^\star \cdot (s^\star - s) \geqslant 0 \quad \text{a.e. on } \omega,$$

and from (4.12) we deduce that

$$\hat{k}|s^\star - s| - \nu\tau^\star \cdot (s^\star - s) = 0 \quad \text{a.e. on } \omega. \tag{4.17}$$

If $\nu|\tau^\star| = \hat{k}$, then from (4.17) we have $\nu|\tau^\star| \cdot |s^\star - s| = \nu\tau^\star \cdot (s^\star - s)$ a.e. on ω; then $\cos(s^\star - s, \nu\tau^\star) = 1$, which implies the existence of $\lambda \geqslant 0$ such that $s^\star - s = \lambda\nu\tau^\star$. And if $\nu|\tau^\star| < \hat{k}$, then from (4.17) we have

$$\hat{k}|s^\star - s| - \nu\tau^\star \cdot (s^\star - s) = 0 \geqslant \left(\hat{k} - |\nu\tau^\star|\right)|s^\star - s| \quad \text{a.e. on } \omega,$$

whence $s^\star - s = 0$ a.e. on ω; then (4.9) follows. □

Theorem 4.3. *Under the same hypothesis of Theorem* 3.2*, and assuming that $\hat{f}$ is a function of x only, we have*

$$\frac{h^2}{2}\nabla p^\star(x) + \nu s^\star(x) + \nu h\tau^\star(x) - \frac{h^2}{2}\hat{f}(x) = 0 \quad \textit{a.e. in } \omega, \tag{4.18}$$

$$\int_\omega \left(h^2\tau^\star(x) + 4hs^\star(x)\right)\nabla\varphi(x)\,dx = 6\int_{\partial\omega} \varphi(x)\tilde{g}(x)\cdot n, \quad \forall\varphi \in H^1(\omega). \tag{4.19}$$

Proof. Integrate twice (4.2) between 0 and y, we obtain

$$\nu u_i^\star(x, y) = \frac{y^2}{2}\frac{\partial p^\star(x)}{\partial x_i} + \nu u_i^\star(x, 0) + \nu y\frac{\partial u_i^\star(x, 0)}{\partial y} - \frac{y^2}{2}\hat{f}_i(x),$$

and as $u_i^\star(x, h) = 0$, (4.18) follows. On the other hand, taking the average of the preceding expression we have

$$\begin{aligned} h\nu\tilde{u}_i^\star(x) &= \int_0^{h(x)} \nu u_i^\star(x, y)\,dy \\ &= \frac{h^3}{6}\frac{\partial p^\star(x)}{\partial x_i} + \nu h u_i^\star(x, 0) + \nu\frac{h^2}{2}\frac{\partial u_i^\star(x, 0)}{\partial y} - \frac{h^3}{6}\hat{f}_i(x). \end{aligned} \tag{4.20}$$

Otherwise, for all φ in $H^1(\omega)$, we have from (3.2),

$$\begin{aligned} \int_\Omega \varphi\,\mathrm{div}(\hat{u}^\varepsilon)\,dx\,dy = 0 &= \int_\omega \varphi(x)\int_0^h \left(\sum_{i=1}^2 \frac{\partial\hat{u}_i^\varepsilon}{\partial x_i} + \frac{\partial\hat{u}_3^\varepsilon}{\partial y}\,dx\right)dy \\ &= \int_\omega \varphi(x)\sum_{i=1}^2\left(\frac{\partial(h\tilde{\hat{u}}_i^\varepsilon)}{\partial x_i} + \hat{u}_3^\varepsilon(x, h) - \hat{u}_3^\varepsilon(x, 0)\right)dx, \end{aligned}$$

then as $\hat{u}_3^\varepsilon = 0$ on $\partial\Omega = \bar{\omega} \cup \bar{\Gamma}_1 \cup \bar{\Gamma}_L$, we have

$$\int_\omega \varphi(x) \sum_{i=1}^{2} \frac{\partial(h\tilde{\hat{u}}_i^\varepsilon)}{\partial x_i}\, dx = 0,$$

where

$$\tilde{\hat{u}}_i^\varepsilon(x) = \frac{1}{h(x)} \int_0^{h(x)} \hat{u}_i^\varepsilon(x, y)\, dy, \quad \forall x \in \omega,$$

and

$$\tilde{g}_i(x) = \int_0^{h(x)} \hat{g}_i(x, y)\, dy = h(x)\tilde{\hat{u}}_i^\varepsilon(x), \quad \forall x \in \partial\omega.$$

Using Green's formula we have

$$\sum_{i=1}^{2} \int_\omega h\tilde{\hat{u}}_i^\varepsilon \frac{\partial\varphi}{\partial x_i}\, dx = \sum_{i=1}^{2} \int_{\partial\omega} h\tilde{\hat{u}}_i^\varepsilon \varphi \cdot \cos(n, x_i) = \sum_{i=1}^{2} \int_{\partial\omega} \tilde{g}_i(x)\varphi \cdot \cos(n, x_i)$$

as $\hat{u}_i^\varepsilon \rightharpoonup u_i^\star$ in V_y then in $L^2(\omega)$, therefore $\tilde{\hat{u}}_i^\varepsilon \rightharpoonup \tilde{u}_i^\star$ in $L^2(\omega)$, and as $\partial\omega \subset \partial\Omega$, we deduce

$$\sum_{i=1}^{2} \int_\omega h\tilde{u}_i^\star \frac{\partial\varphi}{\partial x_i}\, dx = \sum_{i=1}^{2} \int_{\partial\omega} \varphi(x)\tilde{g}_i(x) \cos(n, x_i), \quad \forall\varphi \in H^1(\omega). \tag{4.21}$$

From (4.20) we have

$$\int_\omega \left(\frac{h^3}{6\nu} \nabla p^\star + hs^\star + \frac{h^2}{2}\tau^\star - \frac{h^3}{6\nu}\hat{f} \right) \nabla\varphi\, dx = \int_{\partial\omega} \varphi\tilde{g} \cdot n. \tag{4.22}$$

Then using (4.18) and (4.22), we obtain the weak formulation of the Reynolds equation

$$\int_\omega \left(\frac{h^3}{12\nu} \nabla p^\star - \frac{h}{2}s^\star - \frac{h^3}{12\nu}\hat{f} \right) \nabla\varphi\, dx = \int_{\partial\omega} \varphi\tilde{g} \cdot n. \tag{4.23}$$

Using once again (4.18) and (4.23) we get (4.19). $\square$

5. Study of the uniqueness

In this section, we will give another formulation of the limit inequalities for $s^\star$ and $\tau^\star$ on ω which enables us to express $s^\star$ as a solution of a variational inequality of the second kind with a convenient decomposition. The basic idea is that we have three unknowns $s^\star$, $\tau^\star$, and $\nabla p^\star$ and three relations (4.18), (4.19), and (4.8). A test function in (4.19) appears only to be a gradient function. So it is only possible to control the "gradient" part of $s^\star$ and $\tau^\star$ by this equation which is obtained by a slightly modified version of the well-known decomposition of $L^2(\omega)^2$, due to the nonconstant $h(x)$ coefficients.

Lemma 1. *Let h in $L^\infty(\omega) \cap H^1(\omega)$ such that $h \geqslant \alpha > 0$. Every function ψ in $(L^2(\omega))^2$ has the following orthogonal decomposition*:

$$\psi = h^2 \nabla\varphi + h^{-1}\operatorname{curl}(\theta), \tag{5.1}$$

where φ in $H^1(\omega)/\mathbf{R}$ is the only solution of the problem

$$\int_\omega h^3 \nabla\varphi \nabla\mu \, dx = \int_\omega h\psi \nabla\mu \, dx, \quad \forall \mu \in H^1(\omega), \tag{5.2}$$

and θ in $H_0^1(\omega)$ is the only solution of the problem

$$\int_\omega \operatorname{curl}(\theta)\operatorname{curl}(\xi)\, dx = \int_\omega (h\psi - h^3\nabla\varphi)\operatorname{curl}(\xi)\, dx, \quad \forall \xi \in H_0^1(\omega). \tag{5.3}$$

Proof. As h in $L^\infty(\omega)$, for all ψ in $(L^2(\omega))^2$, we have $h\psi$ in $(L^2(\omega))^2$, following [11, Theorem 3.2], the Neumann's problem (5.2) has a unique solution φ in $H^1(\omega)/\mathbf{R}$. This solution φ satisfies $\nabla(h\psi - h^3\nabla\varphi) = 0$ in $H^{-1}(\omega)$. Hence $h\psi - h^3\nabla\varphi$ is a divergence-free vector of $H(\operatorname{div}, \omega)$. Moreover, Green's formula applied to (5.2) yields

$$0 = \int_\omega (h\psi - h^3\nabla\varphi)\nabla\mu \, dx = \int_{\partial\omega} (h\psi - h^3\nabla\varphi)\cdot n\mu, \quad \forall \mu \in H^1(\omega),$$

implying that $(h\psi - h^3\nabla\varphi)\cdot n = 0$ in $H^{-1/2}(\partial\omega)$. Whence $h\psi - h^3\nabla\varphi$ lies in the space $H = \{v \in (L^2(\omega))^2\colon \operatorname{div}(v) = 0,\ v\cdot n = 0\}$. Moreover, as ω is connected, we deduce, from [11, Theorem 3.1 and its corollary], that the space H is characterized by $H = \{\operatorname{curl}(\mu)\colon \mu \in H_0^1(\omega)\}$, and the mapping curl is an isomorphism from $H_0^1(\omega)$ onto H. So there exists a unique stream function θ in $H_0^1(\omega)$ of $h\psi - h^3\nabla\varphi$ satisfying (5.1) and (5.3). $\square$

Theorem 5.1. *Let h in $L^\infty(\omega) \cap H^1(\omega)$. Under the same hypothesis of Theorem 3.2, $s^\star$ is uniquely given by $s^\star = h^2\nabla C + h^{-1}\operatorname{curl}(D)$, where $U = (C, D)$ is the unique solution of the following variational problem*: *Find U in $H^1(\omega) \times H_0^1(\omega)$ such that*

$$a(U, \phi - U) + J(\phi) - J(U) \geqslant \mathcal{L}(\phi - U), \quad \forall \phi = (\varphi, \theta) \in H^1(\omega) \times H_0^1(\omega), \tag{5.4}$$

where

$$a(U, \phi) = \int_\omega 4\nu h^3 \nabla C \nabla\varphi \, dx + \int_\omega \nu h^{-3}\operatorname{curl}(D)\operatorname{curl}(\theta)\, dx,$$

$$J(\phi) = \int_\omega \hat{k}\big(\big|h^2\nabla\varphi + h^{-1}\operatorname{curl}(\theta) - s\big|\big)\, dx,$$

$$\mathcal{L}\phi = \frac{1}{2}\int_\omega \hat{f}\operatorname{curl}(\theta)\, dx + \int_{\partial\omega} 6\nu\tilde{g}\cdot n\varphi.$$

Proof. From (4.8) and the orthogonal decomposition of ψ, we have

$$\int_\omega \hat{k}\big(\big|h^2\nabla\varphi + h^{-1}\operatorname{curl}(\theta) + s^\star - s\big| - |s^\star - s|\big)\,dx$$
$$\geqslant \int_\omega \nu\tau^\star h^2\nabla\varphi + \int_\omega \nu\tau^\star h^{-1}\operatorname{curl}(\theta)\,dx, \quad \forall(\varphi,\theta)\in H^1(\omega)\times H_0^1(\omega), \tag{5.5}$$

and from (4.19), we have

$$\int_\omega \nu h^2\tau^\star\nabla\varphi = -\int_\omega 4\nu h s^\star\nabla\varphi + \int_{\partial\omega} 6\nu\tilde{g}\cdot n\varphi, \quad \forall\varphi\in H^1(\omega), \tag{5.6}$$

then from (5.5) and (5.6), we have for all (φ,θ) in $H^1(\omega)\times H_0^1(\omega)$,

$$\int_\omega \hat{k}\big(\big|h^2\nabla\varphi + h^{-1}\operatorname{curl}(\theta) + s^\star - s\big| - |s^\star - s|\big)\,dx$$
$$\geqslant -\int_\omega 4\nu h s^\star\nabla\varphi + \int_{\partial\omega} 6\nu\tilde{g}\cdot n\varphi + \int_\omega \nu\tau^\star h^{-1}\operatorname{curl}(\theta)\,dx. \tag{5.7}$$

Now as $s^\star$ in $(L^2(\omega))^2$, we can use its orthogonal decomposition as $s^\star = h^2\nabla C + h^{-1}\operatorname{curl}(D)$, then we deduce for all (φ,θ) in $H^1(\omega)\times H_0^1(\omega)$,

$$\int_\omega \hat{k}\big|h^2\nabla\varphi + h^{-1}\operatorname{curl}(\theta) + h^2\nabla C + h^{-1}\operatorname{curl}(D) - s\big|\,dx$$
$$-\int_\omega \hat{k}\big|h^2\nabla C + h^{-1}\operatorname{curl}(D) - s\big|\,dx$$
$$\geqslant -\int_\omega 4\nu h^3\nabla C\nabla\varphi - 4\nu\int_\omega \operatorname{curl}(D)\nabla\varphi + \int_{\partial\omega} 6\nu\tilde{g}\cdot n\varphi + \int_\omega \nu\tau^\star h^{-1}\operatorname{curl}(\theta)\,dx. \tag{5.8}$$

Using (4.18) we have

$$\int_\omega \nu\tau^\star h^{-1}\operatorname{curl}(\tau^\star)\,dx = \int_\omega \left(-\frac{1}{2}\nabla p^\star - \frac{\nu}{h^2}s^\star + \frac{1}{2}\hat{f}\right)\operatorname{curl}(\theta)\,dx,$$

then

$$\int_\omega \nu\tau^\star h^{-1}\operatorname{curl}(\theta)\,dx = -\int_\omega \frac{1}{2}\operatorname{curl}(\theta)\nabla p^\star\,dx - \int_\omega \nu\operatorname{curl}(\theta)\nabla C\,dx$$
$$-\int_\omega \nu h^{-3}\operatorname{curl}(D)\operatorname{curl}(\theta)\,dx + \frac{1}{2}\int_\omega \hat{f}\operatorname{curl}(\theta)\,dx.$$

Using Green's formula and that θ in $H_0^1(\omega)$, we have

$$\int_\omega \operatorname{curl}(\theta)\nabla p^\star\,dx = -\big\langle p^\star, \operatorname{div}\big(\operatorname{curl}(\theta)\big)\big\rangle + \int_{\partial\omega}\operatorname{curl}(\theta)\cdot n p^\star\,dx = 0,$$

by the same argument we also have

$$\int_\omega \nu \operatorname{curl}(\theta)\nabla C = \int_\omega \operatorname{curl}(D)\nabla\varphi = 0.$$

Then from (5.8), $U = (C, D)$ satisfies for all $\phi = (\varphi, \theta)$ in $H^1(\omega) \times H_0^1(\omega)$,

$$\begin{aligned}
&\int_\omega \left\{4\nu h^3 \nabla C \nabla\varphi + \nu h^{-3} \operatorname{curl}(D) \operatorname{curl}(\theta)\right\} dx \\
&\qquad + \int_\omega \hat{k}\left|h^2\nabla(\varphi + C) + h^{-1}\operatorname{curl}(\theta + D) - s\right| dx \\
&\qquad - \int_\omega \hat{k}\left|h^2\nabla C + h^{-1}\operatorname{curl}(D) - s\right| dx \\
&\quad \geqslant \int_\omega \frac{1}{2}\hat{f} \operatorname{curl}(\theta)\, dx + \int_{\partial\omega} 6\nu\tilde{g}\cdot n\varphi,
\end{aligned}$$

taking $\tilde{\varphi} = \varphi + C$ and $\tilde{\theta} = \theta + D$ we deduce the variational inequality 5.4. As the bilinear form $a(\cdot,\cdot)$ is continuous and coercive, the functional J is convex, proper and continuous, and the linear form $\mathcal{L}$ is continuous, the existence and uniqueness of (C, D) in $H^1(\omega) \times H_0^1(\omega)$ follows, and implies the existence and uniqueness of $s^\star$ in $(L^2(\omega))^2$. $\square$

Theorem 5.2. *Under the same hypothesis of Theorem* 5.1*, there exists a unique solution $p^\star$ in $H^1(\omega)$ satisfying the weak formulation of the Reynolds equation* (4.23)*.*

Proof. From Theorem 5.1, $s^\star$ is unique in $(L^2(\omega))^2$, then the uniqueness of $p^\star$ follows from (4.23). $\square$

Remark 3. $\tau^\star$ is then unique from the uniqueness of $p^\star$ and $s^\star$ using (4.18).

Remark 4. In the case where $\omega =]a, b[$ is a bounded domain of $\mathbf{R}$, the system (2.1)–(2.6), can be viewed as a thin film lubrication problem describing flow of fluids in an infinitely long journal bearing where a cross section is given by $\Omega^\varepsilon = \{(x, x') \in \mathbf{R}^2\colon\ x \in]a, b[,\ 0 < x' < \varepsilon h(x)\},\ \varepsilon > 0$. The results of Sections 1–5 remain valid and it is possible to get $s^\star$ as the unique solution of a variational inequality of the second kind. Moreover, it will be proved in the following theorem that $\tau^\star$, the limiting stress on ω, can be obtained directly as the unique solution of a variational inequality of the first kind. Let us introduce the closed convex $\mathcal{K} = \{\psi \in L^2(\omega)\colon\ |\psi| \leqslant \hat{k}/\nu,\ \text{a.e. in } \omega\}$.

Theorem 5.3. *Under the same hypothesis of Theorem* 3.2*, $\tau^\star$, the limiting stress on $\omega =]a, b[$, is the unique solution in $\mathcal{K}$ of the variational inequality*

$$\int_a^b (h^2\tau^\star + 4hs)(\psi - \tau^\star)\, dx - \tilde{g}(b)\int_a^b (\psi - \tau^\star)\, dx \geqslant 0, \quad \forall\psi \in \mathcal{K}. \tag{5.9}$$

Proof. As $\omega =]a, b[\subset \mathbf{R}$, then for any ψ in $L^2(\omega)$ there exists φ in $H^1(\omega)$, such that $\varphi(x) = \varphi(a) + \int_a^x \psi(t)\,dt$, then from (4.19) we get, using the $\tilde{g}(b) = \tilde{g}(a)$ mass flow conservation property

$$\int_a^b (h^2\tau^\star + 4hs^\star)\psi\,dx = 6\tilde{g}(b)\bigl\{\varphi(b) - \varphi(a)\bigr\} = 6\tilde{g}(b)\int_a^b \psi(t)\,dt, \quad \forall\psi \in L^2(\omega). \tag{5.10}$$

As $\tau^\star$ in $L^2(\omega)$, we get

$$\int_a^b (h^2\tau^\star + 4hs^\star)(\psi - \tau^\star)\,dx = 6\tilde{g}(b)\int_a^b (\psi - \tau^\star)\,dx, \quad \forall\psi \in L^2(\omega). \tag{5.11}$$

Now, we compute

$$\int_a^b (h^2\tau^\star + 4hs)(\psi - \tau^\star)\,dx = \int_a^b (h^2\tau^\star + 4hs^\star)(\psi - \tau^\star)\,dx + \int_a^b 4h(s - s^\star)(\psi - \tau^\star)\,dx.$$

Using now (4.8) and (4.9) we have

$$\int_a^b 4h(s^\star - s)(\psi - \tau^\star)\,dx = \int_{\nu|\tau^\star| = \hat{k}} 4h\lambda\tau^\star(\psi - \tau^\star)\,dx.$$

As $h\lambda \geqslant 0$, we have for all ψ in $\mathcal{K}$,

$$\int_{\nu|\tau^\star| = \hat{k}} 4h\lambda\tau^\star(\psi - \tau^\star)\,dx = \int_{\nu|\tau^\star| = \hat{k}} 4h(x)\lambda(\tau^\star\psi - \tau^\star \cdot \tau^\star)\,dx = \int_{\nu|\tau^\star| = \hat{k}} 4h\lambda\left(\tau^\star\psi - \left(\frac{\hat{k}}{\nu}\right)^2\right)dx \leqslant 0. \tag{5.12}$$

Thus from (5.11) and (5.12), (5.9) follows. □

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